

Serie n°1 – Week of September 9th

Algebraic structures, $\mathbb{N}$ and $\mathbb{Z}$
Crystals and Symmetries

Exercise 1: Point Group 2/m

We consider a three dimensional Euclidean space with an orthonormal basis $\mathcal{B}_{(0,x,y,z)}$, and a set of four symmetry operators:

- The identity (1) (or 1-fold rotation):
For a point $P(x, y, z)$ in $\mathcal{B}_{(0,x,y,z)}$, $1(P) = P$.
- A 2-fold rotation (2) (rotation of angle π) around the z axis:
For a point $P(x, y, z)$ in $\mathcal{B}_{(0,x,y,z)}$, $2(P) = P'$ with $P'(-x, -y, z)$
- A mirror symmetry (m) across the $z = 0$ plane ((x, y) plane):
For a point $P(x, y, z)$ in $\mathcal{B}_{(0,x,y,z)}$, $m(P) = P'$ with $P'(x, y, -z)$
- The inversion operation ($\bar{1}$) through the origin O:
For a point $P(x, y, z)$ in $\mathcal{B}_{(0,x,y,z)}$, $\bar{1}(P) = P'$ with $P'(-x, -y, -z)$

We want to show that these four operators form a point symmetry group, noted 2/m in the Hermann-Mauguin notation.

1a. Fixed point:

Can you find a point that is left unchanged by all 4 symmetry operators ?

We use the symbol “o” between two arbitrary operators f and g the same way it is used for functions: if $g(P) = P'$, $fog(P) = f(g(P)) = f(P') = P''$.

The inverse f^{-1} of a symmetry operator f is then defined as $\forall P, fof^{-1}(P) = P$, or $fof^{-1} = 1$.

1b. Show that for all symmetry operators described above, the operator is equal to its inverse.

1c. Show that:

- (i) $\forall P(x, y, z), 2om(P) = mo2(P) = \bar{1}(P)$
- (ii) $\forall P(x, y, z), \bar{1}o2(P) = 2o\bar{1}(P) = m(P)$
- (iii) $\forall P(x, y, z), \bar{1}om(P) = mo\bar{1}(P) = 2(P)$

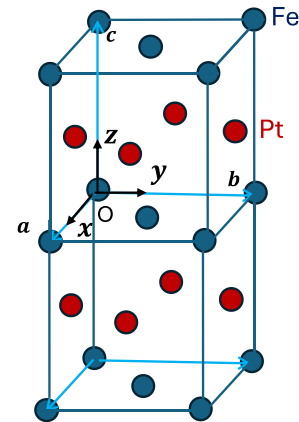
1d. Associativity:

Deduce from 5c that $2o(mo\bar{1}) = (2om)o\bar{1}$ and $mo(\bar{1}o2) = (mo\bar{1})o2$.

Exercise 2: Iron Platinum

Iron-Platinum is a magnetic material that crystallizes in the structure shown. In this structure, we suppose that the Iron atoms sits at the corner of a cube of edge a , as well as at the center of the top and bottom cube faces. Pt atoms sits at the center of the 4 other faces.

The crystal can be formed by translating this cube along the $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ orthogonal vectors. On the schematic, two cubes are represented.



2a. How many atoms of Fe and Pt can be counted in a single cube ? Deduce the chemical structure of this material ?

2b.

- (i) Show schematically that the crystal structure is actually tetragonal primitive, and find its lattice parameters as a function of a .
- (ii) What is the motif ?

2c. The FePt crystal has the space group symmetry 123 , which contains 16 symmetry operations:

- What are the two roto-inversions ? Specify the axis and the n -fold nature.
- Why there is no roto-inversion $\bar{2}$ listed ?
- There are 5 2-fold rotations listed: can you identify the axis ?
- There are 2 4-fold rotations listed: can you identify the axis ?
- There are 5 mirror symmetries listed: can you identify them ? (across the (x,y) plane for example)
- What are the two remaining symmetry operations not discussed above ?

Exercise 3: Subgroups

We consider the set of real Polynomial functions of the form:

$$\text{For } x \in \mathbb{R}, n \in \mathbb{N}, P(x) = \sum_{k=0}^n a_k x^k = a_0 + a_1 x + \dots + a_n x^n \text{ with } \forall k \in \mathbb{N}, a_k \in \mathbb{R}.$$

The degree of the polynomial is defined as the highest value of the exponent k for which $a_k \neq 0$.

3a.

- (i) Show that the set of all polynomials associated to the addition operation form a group $G_{[X]}$.

- (ii) Would it still be a group if we replaced the addition operation with the multiplication ?

Given a group G under an operation (often called binary operation) $*$, a subset H of G is called a **subgroup** of G if H also forms a group under the operation $*$.

3b. Show that the subset of Polynomials with even coefficients form a subgroup of $G_{[X]}$.
Hint: you can consider that 0 is an even number.

3c. Consider the subset of polynomials with an even degree. Does it form a subgroup of $G_{[X]}$?

3d. Consider the subset of polynomials with at least one real root (i.e. one real number x_0 for which $P(x_0) = 0$). Does it form a subgroup of $G_{[X]}$?

3e. (more abstract extra question, just for fun)

Let's consider a group G for an operation $*$, and two subgroups A and B of G . Show that $A \cup B$ is a subgroup of G if and only if $A \subseteq B$ or $B \subseteq A$.

Hint on notations: $A \cup B$, or A union B , is a set that gathers all the elements of both groups.

$A \subseteq B$, or A included in B means that all the elements of A also belong to B .

Exercise 4: The Fibonacci sequence

The Fibonacci sequence first appears in his Liber Abaci (The Book of Calculation, 1202) where he calculates the growth of rabbit populations. Assume that:

- A newly born breeding pair of rabbits are put in a field at month 1;
- Each breeding pair mates at the age of one month, and at the end of their second month they always produce another pair of rabbits;
- Rabbits never die, but continue breeding forever.

At month $n \in \mathbb{N}$, we call F_n the number of pairs of rabbits ($F_0 = 0$ & $F_1 = 1$).

4a. How many pairs of rabbit will there be at months two, three and four?

4b. For $n \geq 2$, explain why: $F_n - F_{n-1} = F_{n-2}$

4c. For $n \in \mathbb{N}$, find the two real solutions $\varphi > 0$ and $\psi < 0$ of the equation:

$$x^{n+2} = x^{n+1} + x^n.$$

4d. We define the sequence $u_n = a\varphi^n + b\psi^n$, with $(a, b) \in \mathbb{R}^2$. Show that:

$$\forall n \in \mathbb{N}, u_{n+2} = u_{n+1} + u_n$$

4e. Express $(a, b) \in \mathbb{R}^2$ as a function of φ and ψ such that $u_0 = 0$ and $u_1 = 1$

4f. Conclude that $\forall n \in \mathbb{N}, F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$